MTH 203 - Quiz 1 Solutions

1. If G is a group in which $g^2 = 1$, for every non-trivial $g \in G$, then show that G is abelian. Give a non-trivial example of such a group G.

   Solution. For arbitrary $a, b \in G$, by hypothesis we have

   $$(ab)(ab) = a^2b^2,$$

   which upon right and left cancellation yields

   $$ab = ba.$$

   Thus, it follows that G is abelian. As we saw in class, the Klein-4 group U_8 is a group of order 4 in which each nontrivial elements has order 2.

2. Let G be a group and $H < G$ such that $[G : H] = 2$. Then show that $H \lhd G$.

   Solution. Since $[G : H] = 2$, we have $G/H = \{H, kH\}$, for any $k \in G \setminus H$. Since $G = H \sqcup Hk$, the right coset Hk can either equal the coset H or the coset Hk. If $Hk = H$, then it follows that $k \in H$ (why?), which contradicts the choice of k. Thus, it follows that $Hk = kH$ for every $k \in G \setminus H$, and this also holds trivially for every $k \in H$. Therefore, $Hk = kH$ for every $k \in G$, and so by assertion 3.1 (iii) of Lesson Plan it follows that $H \lhd G$.